A STUDY ON DISTRIBUTIONAL SYSTEMS

Assoc. Prof. Liviu Popescu Ph. D
University of Craiova
Dept. of Applied Mathematics
Craiova, Romania
Lecturer Radu Criveanu Ph. D
University of Craiova
Faculty of Economics and Business Administration
Craiova, Romania

Abstract: In this paper we continue the study of distributional systems (drift less control affine systems) with quadratic cost and no constant rank of distribution. We will use the Pontyagin Maximum Principle and the constants variation method for nonhomogeneous second order differential equations to find the general solution.

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1. INTRODUCTION

This paper continues the study of distributional systems started by the first author in [3], [4], [5], [6], [7], [8], [9]. It is well known that the optimal solution of a distributional system (see [1]) is provided by Pontryagin's Maximum Principle: that is, the curve $c(t)=(x(t),u(t))$ is an optimal trajectory if there exists a lifting of $x(t)$ to the dual space $(x(t),p(t))$ satisfying the Hamilton's equations.

We have to remark that the optimal solutions of our system are the geodesics in the so-called sub-Riemannian geometry (see [2]). We find the general solution of the control problem using the constants variation method for nonhomogeneous second order differential equations. We are in the case of strong bracket generating distribution (i.e. the vector fields of distribution and the first iterated Lie brackets generate the entire space R^3). The well-known Chow's theorem quarantees that the system is controllable, that is the system can be brought from any state x_1 to any other state x_2. The second section is introductory in the theory of distributional systems.

2. DISTRIBUTIONAL SYSTEMS

We consider a distributional systems in the space R^n in the following form (see [2])

$$\dot{X}(t)=\sum_{i=1}^{m}u^i(t)X_i(x(t)), \qquad (1)$$

where X_i, $i=\overline{1,m}$ are vector fields in the space R^n and the controls $u=(u_1,u_2,...,u_m)$ take values in an open subset $\Omega \subset R^n$. The vector fields X_i generate a distribution $D \subset R^n$ such that the rank of D is not necessarily constant.

An optimal control problem consists of finding those trajectories of the distributional system which connect any two points x_0 and x_1 from R^n, while minimizing the cost

$$\min_{u(.)} \int_I F(x(t),u(t))dt, \tag{2}$$

where F is the quadratic cost $F=\sqrt{\sum_{i=1}^{m} u_i^2(t)}$ on D.

The optimal solutions are obtained by integrating the system (1). If the distribution D is bracket generating (i.e. the vector fields of D and iterated Lie brackets generate the entire space R^n), then by a well-known theorem of Chow the system (1) is controllable, that is for any two points x_0 and x_1 there exists an optimal trajectory which connects these points.

The necessary conditions for a trajectory to be optimal are given by Pontryagin Maximum Principle. The Hamiltonian has the form [1]

$$H(x,p,u) = < p, \dot{X} > - L(x,u),, \tag{3}$$

where p is the momentum variable on the dual space and $L=\frac{1}{2}F^2$ is the Lagrange function. The maximixation condition with respect to the control variables u, namely

$$H(x(t),p(t),u(t)) = \max_v H(x(t),p(t),v),$$

leads to the equations

$$\frac{\partial H(x,p,u)}{\partial u} = 0, \tag{4}$$

and the optimal trajectories satisfy the Hamilton's equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}. \tag{5}$$

3. Application

Let us consider a distributional system in the space R^3 of the form

$$\dot{X}(t) = u^1 X_1 + u^2 X_2 + u^3 X_3, \tag{6}$$

with the vectors X_1, X_2, X_3 given by

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}$$

and minimizing the functional

$$\min_{u(.)} \int_I F(u(t))dt$$

where $F = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2}$ is the quadratic cost.

In the following we assume that the trajectories are parameterized by arclength and starting from the origin. The distribution *D* is generated by the vectors X_1, X_2, X_3 and the rank of distribution is not constant

$$rankD = \begin{cases} 3 & if \quad x \neq 0, \\ 2 & if \quad x = 0. \end{cases}$$

In the canonical base of the three dimensional space R^3, $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$ the vectors X_1, X_2, X_3 have the following expressions

$$X_1 = \frac{\partial}{\partial x}, \ X_2 = \frac{\partial}{\partial y}, \ X_3 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

Using the formula

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X,$$

the Lie brackets of the vectors X_1, X_2, X_3 are given by

$$[X_1, X_2] = 0,$$

$$[X_1, X_3] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] + \left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial z}\right] = \frac{\partial}{\partial z} = X_4 \notin D,$$

$$[X_2, X_3] = \left[\frac{\partial}{\partial y}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right] = \left[\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right] + \left[\frac{\partial}{\partial y}, x\frac{\partial}{\partial z}\right] = 0,$$

$$[X_1, X_4] = 0, [X_2, X_4] = 0, [X_3, X_4] = 0.$$

We obtain that $X_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and the vectors $\{X_1, X_2, X_3, X_4 = [X_1, X_3]\}$ generate the entire space R^3. It result that the distribution is not integrable (nonholonomic), but is strong bracket generating, that is the vector fields of the distribution and the first iterated Lie brackets span the entire space R^3.

From the relation (6) we obtain

$$\dot{X}(t) = u^1 X_1 + u^2 X_2 + u^3 X_3 = \begin{pmatrix} u^1 \\ u^2 + u^3 \\ u^3 x \end{pmatrix},$$

and it results

$$\begin{cases} \dot{x} = u^1 \\ \dot{y} = u^2 + u^3 \\ \dot{z} = u^3 x \end{cases}$$

Considering the Lagrangian function $L = \frac{1}{2}F^2 = \frac{1}{2}\left(\left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2\right)$ the Pontryagin maximum principle leads to the following Hamiltonian function $H(x,p,u) = < p, \dot{X} > -L(u) = p_1 u^1 + p_2\left(u^2 + u^3\right) + p_3 u^3 x - \frac{1}{2}\left(\left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2\right).$

The conditionn (4) yields the equations

$$\frac{\partial H}{\partial u^1} = 0 \Leftrightarrow p_1 - u^1 = 0 \Leftrightarrow p_1 = u^1,$$

$$\frac{\partial H}{\partial u^2} = 0 \Leftrightarrow p_2 - u^2 = 0 \Leftrightarrow p_2 = u^2,$$

$$\frac{\partial H}{\partial u^3} = 0 \Leftrightarrow p_2 + p_3 x - u^3 = 0 \Leftrightarrow p_2 + p_3 x = u^3.$$

and the Hamiltonian has the form

$$H(x,p) = (p_1)^2 + (p_2)^2 + (p_2 + p_3 x)^2 - \frac{1}{2}\left((p_1)^2 + (p_2)^2 + (p_2 + p_3 x)^2\right) =$$

$$= \frac{1}{2}(p_1)^2 + (p_2)^2 + \frac{1}{2}(p_3)^2 x^2 + p_2 p_3 x$$

.

The Hamilton's equations (5) lead to the following system of partial differential equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_1} = p_1,$$

$$\frac{dy}{dt} = \frac{\partial H}{\partial p_2} = 2p_2 + p_3 x,$$

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_3} = p_3 x^2 + p_2 x,$$

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x} = -(p_3)^2 x - p_2 p_3,$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial y} = 0 \Rightarrow p_2 = a,$$

$$\frac{dp_3}{dt} = -\frac{\partial H}{\partial z} = 0 \Rightarrow p_3 = b,$$

where a,b are real constants. It results

$\frac{d^2x}{dt^2} = \frac{dp_1}{dt} = -b^2x - ab \Rightarrow \frac{d^2x}{dt^2} + b^2x = -ab$. This is a second order nonhomogeneous differential equation. We consider the corresponding homogeneous differential equation $\frac{d^2x}{dt^2} + b^2x = 0$, with the characteristic equation $r^2 + b^2 = 0$. The solutions $r_{1,2} = \pm bi$ lead to the general solution of the second order homogeneous differential equation

$$x(t) = c_1 \cos bt + c_2 \sin bt,$$

where $c_1, c_2 \in R$.

Using the constants variation method, we are looking for a particular solution of a nonhomogeneous differential equation of the form $x(t) = \alpha_1(t)\cos bt + \alpha_2(t)\sin bt$, satisfying the equations

$$\alpha_1' \cos bt + \alpha_2' \sin bt = 0 \qquad (7)$$

$$-\alpha_1' b \sin bt + \alpha_2' b \cos bt = -ab \qquad (8)$$

The equation (7) multiplied by $b \sin bt$ plus equation (8) multiplied by $\cos bt$ yields

$$\alpha_2' = -a\cos bt \text{ and it results } \alpha_2 = -\frac{a}{b}\sin bt.$$

Also, the equation (7) multiplied by $b\cos bt$ minus equation (8) multiplied by $\sin bt$ yields

$$\alpha_1' = a\sin bt \text{ and it results } \alpha_1 = -\frac{a}{b}\cos bt.$$

The particular solution of the second order nonhomogeneous differential equation has the form

$$x(t) = -\frac{a}{b}\cos^2 bt - \frac{a}{b}\sin^2 bt = -\frac{a}{b},$$

and it results the general solution

$$x(t) = c_1 \cos bt + c_2 \sin bt - \frac{a}{b}.$$

But $x(0) = 0$ and it results $c_1 = \frac{a}{b}$, that is the general solution has the form

$$x(t) = \frac{a}{b}(\cos bt - 1) + c\sin bt, \qquad c \in R \qquad (9)$$

From the equation $\frac{dy}{dt} = 2a + bx$ we obtain

$$\frac{dy}{dt} = 2a + a(\cos bt - 1) + bc\sin bt$$

and it results

$$\frac{dy}{dt} = a + a\cos bt + bc\sin bt$$

with the solution

$$y(t) = at + \frac{a}{b}\sin bt - c\cos bt + d, \qquad d \in R.$$

But $y(0) = 0$ and it results $d = c$, which give the solution

$$y(t) = at + \frac{a}{b}\sin bt + c(1 - \cos bt). \tag{10}$$

The equation $\frac{dz}{dt} = bx^2 + ax$ leads to

$$\frac{dz}{dt} = b\left(\frac{a}{b}(\cos bt - 1) + c\sin bt\right)^2 + a\left(\frac{a}{b}(\cos bt - 1) + c\sin bt\right).$$

By direct computation it results

$$\frac{dz}{dt} = \frac{a^2}{b}\cos^2 bt + bc^2 \sin^2 bt - \frac{a^2}{b}\cos bt + ac\sin 2bt - ac\sin bt,$$

with the solution

$$z(t) = \frac{a^2}{4b^2}\sin 2bt + \frac{a^2 t}{2b} - \frac{c^2}{4}\sin 2bt + \frac{bc^2 t}{2} - \frac{a^2}{b^2}\sin bt - \frac{ac}{2b}\cos 2bt + \frac{ac}{b}\cos bt,$$

or

$$z(t) = \left(\frac{a^2}{4b^2} - \frac{c^2}{4}\right)\sin 2bt - \frac{a^2}{b^2}\sin bt - \frac{ac}{2b}\cos 2bt + \frac{ac}{b}\cos bt + \left(\frac{a^2}{2b} + \frac{bc^2}{2}\right)t + d.$$

But $z(0) = 0$ and it results $d = -\frac{ac}{2b}$ that is the solution has the form

$$z(t) = \left(\frac{a^2}{4b^2} - \frac{c^2}{4}\right)\sin 2bt - \frac{a^2}{b^2}\sin bt - \frac{ac}{2b}(\cos 2bt + 1) + \frac{ac}{b}\cos bt + \left(\frac{a^2}{2b} + \frac{bc^2}{2}\right)t.$$

From the equation $\frac{dp_1}{dt} = -b^2 x - ab$ it results

$$\frac{dp_1}{dt} = -b^2\left(\frac{a}{b}(\cos bt - 1) + c\sin bt\right) - ab,$$

or in the equivalent form

$$\frac{dp_1}{dt} = -ab\cos bt - b^2 c\sin bt,$$

with the solution

$$p_1(t) = -a\sin bt + bc\cos bt.$$

In these conditions the Hamiltonian has the form

$$H = \frac{1}{2}(-a\sin bt + bc\cos bt)^2 + a^2 + \frac{1}{2}b^2 x^2 + abx.$$

By direct computation it results $H = a^2 + \frac{b^2c^2}{2}$. Considering that the optimal trajectories are parametrized by arclength, that is $(u^1)^2 + (u^2)^2 + (u^3)^2 = 1$ we obtain $H = \frac{1}{2}$, which yields $a^2 + \frac{b^2c^2}{2} = \frac{1}{2}$ and we obtain

$$c = \pm\frac{\sqrt{1-2a^2}}{b}, \quad b \neq 0 \quad a \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right],$$

which end the proof.
Finally, the optimal solution has the form

$$x(t) = \frac{a}{b}(\cos bt - 1) + c\sin bt, \quad c = \pm\frac{\sqrt{1-2a^2}}{b},$$

$$y(t) = at + \frac{a}{b}\sin bt + c(1-\cos bt),$$

$$z(t) = \left(\frac{a^2}{4b^2} - \frac{c^2}{4}\right)\sin 2bt - \frac{a^2}{b^2}\sin bt - \frac{ac}{2b}(\cos 2bt + 1) + \frac{ac}{b}\cos bt + \left(\frac{a^2}{2b} + \frac{bc^2}{2}\right)t.$$

4. Conclusions

We have obtained the general solution of a distributional system with quadratic cost and no constant rank of distribution, using the Pontryagin Maximum Principle. The Hamilton's equations lead to a nonhomogeneous second order differential ecuation. We apply the constants variation method in order to find the general solution.

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