CHAOS TESTS FOR TIME SERIES

Dumitru Ciobanu Ph. D Student
University of Craiova
Faculty of Economics and Business Administration
Craiova, Romania

Abstract: Chaos Theory aims to find the underlying order from apparently random data. Every economic process produce one or many time series. Determining if the process is or not chaotic may supply valuable information about how to deal with that process. There is no single test that identify chaos, so to say that a system is chaotic is better to perform more tests.

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1. INTRODUCTION

Chaotic systems are in fact complex deterministic systems with a large number of variables that influence the evolution of the process making it impossible for humans to simulate it and therefore making them seems unpredictable. This, also, makes it impossible to determine the initial state of the system knowing just the final state.

Chaos is met in: solar system dynamics, evolution of populations, the weather, chemical reactions, etc. Also, the economy can be seen as a chaotic system, a factor that brings a huge number of variables is direct involvement of people. The chaos from complex systems is known as chaos deterministic.

Chaotic behavior is indicated by the following features:

- Chaotic processes are nonlinear processes.
- Chaotic processes are deterministic processes that retain their size when immersed in a larger space.
- If chaotic processes there will be strange attractors.
- Chaotic processes are sensitive to initial conditions.

At this point, there is no reliable test to certify the existence of chaos. A working version for detecting nonlinear dynamics and chaotic time series, would be to apply several tests indicating the presence of chaos and avoid erroneous conclusions that can be drawn from a single test (Georgescu, 2012)

In this paper are presented some of the commonly used tests to highlight the chaotic behavior: Phase space reconstruction and embedding dimension, Fractal dimension and Largest Lyapunov Exponent.

2. PHASE SPACE RECONSTRUCTION AND EMBEDDING DIMENSION

Phase space reconstruction and the graphical representation of phase space may reveal the existence of strange attractor which is one of the indicators of chaotic behavior.

Phase space reconstruction is used also by other methodologies such us BDS test and correlation dimension estimation.

We assume that information is available and represented as a discrete univariate time series $\{x_t\}$. It can be assumed that the time series is a one-dimensional projection of a source signal represented by a dynamic state vector of dimension *d*, $\{x_t^d\}$. The dynamic state vector consists of *d* variables appropriate to its size *d* and *t* is the current time in the time series sample.

Transition from one-dimensional time series $\{x_t\}$ to the *d*-dimensional sample coresponding from state space is done using Takens theorem. Takens theorem is a technique for reconstruction of an approximation of the unknown *d*-dimensional vector $\{x_t^d\}$ from state space by delaying and embedding the observed time series $\{x_t\}$ (Takens, 1981). The reconstruction of phase space suggested by Takens has four steps:

- Suppose it's available a time series with *N* observations $\{x_1, x_2, \ldots x_N\}$.
- Is determined an appropriate time delay τ.
- Is determined the embedding dimension *d*.
- The one-dimensional time series is embedded in *d*-dimensional space by constructing the next vectors
  $x_t^d = \left(x_t, x_{t+\tau}, \ldots, x_{t+\tau(d-1)}\right),\ t = 1,2,\ldots, N - \tau(d-1)$.

This approximate reconstruction is the state vector composed by time delays of the time series sample $\{x_t\}$, where τ is the number of time delays and *d* is the embedding dimension of the system. The accurate calculation of *d* and τ guarantees, according to Embedding Theorem (Abarbanel, 1996), that the sequential order of the reconstructed state vector $x_t^d \to x_{t+1}^d$, is topologically equivalent to the state vector generation $x_t \to x_{t+1}$, allowing $\{x_t^d\}$ to represent unambiguously the original source of the observed time series $\{x_t\}$.

2.1 Determining a reasonable time delay - Mutual Information

Accurate determination of the time delay τ ensures that the coordinate vectors $\{x_t^d\}$ from time delayed space states are independent of each other. If is choosen a small value for τ, data from state space are clustered while choosing a too high value for τ results in the disappearance of relations between points and attractor.

The determination of time delay τ can be done using mutual information function (Fraser & Swinney, 1986).

The proposed criterion suggests that the delay to be set to the first local minimum of mutual information function between coordinates.

The relationship with the help which is calculated the mutual information function between two coordinates of $\{x_t^d\}$, for example x_t^d and $x_{t+\tau}^d$ is

$$MI(\tau) = \log_2 \frac{P\left(x_t^d, x_{t+\tau}^d\right)}{P\left(x_t^d\right)P\left(x_{t+\tau}^d\right)}$$

where $P\left(x_t^d, x_{t+\tau}^d\right)$ is the joint probability density function of x_t^d and $x_{t+\tau}^d$.

For two discrete variables X and Y the joint probability distribution is $P_{XY}(x,y)$.

The average mutual information of all coordinates is calculated with:

$$AMI(\tau)=\sum_{x_t^d, x_{t+\tau}^d} P\left(x_t^d, x_{t+\tau}^d\right)\log_2 \frac{P\left(x_t^d, x_{t+\tau}^d\right)}{P\left(x_t^d\right)P\left(x_{t+\tau}^d\right)}.$$

Using the mutual information to determine the time delay is based on the idea that a good choice for time delay shall be designed so that x_t^d provide maximum information about $x_{t+\tau}^d$. Fraser and Swinney states that the best value for the time delay is the smallest value τ for which $AMI(\tau)$ is a local minimum and ensures the independence between the coordinates of multidimensional vector $\{x_t^d\}$.

This value of the time delay can be used for graphical representation of phase space. $AMI(\tau)$ must be calculated using the joint probability distribution $P\left(x_t^d, x_{t+\tau}^d\right)$ for several values of τ. The mutual information can detect non-linear correlations. Graphic representations of the the mutual information $AMI(\tau)$ are reminiscent of simple autocorrelation plots and can highlight any kind of dependencies (Georgescu, 2012).

2.2 Determining the embedding dimension – False Nearest Neighbours

The signal reconstruction in state space requires a dimension to ensure that there will be no overlap of the orbits of the dynamical system. The optimal dimension is obtained by calculating the percentage of false nearest neighbors (FNN) between points in state space.

The number of false nearest neighbors is calculated using the reconstructed state space vectors x_t^d of different embedding dimension *d*, but with a constant number of time delay τ. Is generally accepted that when the percentage of false nearest neighbors drops to zero is reached the minimum size for embedding the original state space system around the attractor, guaranteeing also that the orbit is unique. Calculation of false nearest neighbors requires measuring the distance R_d, defined as the radius between neighboring vectors in consecutive dimensions.

The square of the euclidian distance R_d in dimension *d* is:

$$R_d^{\,2}(t)=\sum_{m=1}^{d}\left(x_{t+\tau(m-1)}-x_{t+\tau(m-1)}^{NN}\right)^2$$

where *t* is the current index of the discrete signal (x_t) and x_t^{NN} is the nearest neighbour (NN) of x_t.

In dimension *d+1* the square of euclidian distance is:

$$R_{d+1}^{\,2}(t)=\sum_{m=1}^{d+1}\left(x_{t+\tau(m-1)}-x_{t+\tau(m-1)}^{NN}\right)^2 = R_d^{\,2}(t)+\left(x_{t+\tau d}-x_{t+\tau d}^{NN}\right)^2.$$

It is considered the criterion according to which a false nearest neighbor is any neighbor for which the change in distance between points in dimension *d* and dimension *d+1* exceeds a heuristic threshold R_{tol}:

$$\sqrt{\frac{R_{d+1}{}^2(t)-R_d{}^2(t)}{R_d{}^2(t)}}=\frac{\left\|x_{t+\tau d}-x_{t+\tau d}^{NN}\right\|}{R_d(t)}>R_{tol}\ .$$

Determination of false nearest neighbors depends on how is changed the distance between vectors in state space in consecutive dimensions. If the distance increases significantly with embeding dimension increment, then vectors are false neighbors, this means that points apparently close because the projection are separate by large distances when the embedding dimension is increased. If the gap remains below a certain threshold, the points in state space are real neighbors resulting from the dynamics of the system. Embedding dimension that accurately represent the system is that which eliminates the most false neighbors resulting in a system whose state space trajectories are positioned according to system dynamics and not due to space reconstruction (Georgescu, 2012).

2.3 Determining the embedding dimension – The Cao's method

To understand how to choose a good dimension *d* for embedding is helpful to understand what is happening geometrically. As the size increases chaotic attractor unfolds. When the attractor is completely unfolded, a sequential path from one point to another will not self intersect. If the embedding dimension *d* is too small, some paths of the projected attractor will self intersect.

Method of false nearest neighbors recognize that where paths of attractor self intersect, two neighboring points are really distant in the embedding space of data series (Georgescu, 2012).

Using this idea, Cao (1997) proposed a method for determining a good embedding dimension.

Let

$$E_1(d)=\frac{E(d+1)}{E(d)}$$

with

$$E(d)=\frac{1}{N-d\tau}\sum_{t=1}^{N-d\tau}\frac{\left\|x_t^{d+1}-x_t^{d+1,NN}\right\|}{\left\|x_t^{d}-x_t^{d,NN}\right\|}$$

and

$$\left\|x_t^{d}-x_t^{d,NN}\right\|=\max_{0\le m\le d-1}\left|x_{t+m\tau}-x_{t+m\tau}^{NN}\right|,$$

where *N* is the length of initial series of data, *d* represent the embedding dimension and superscript *NN* identify the nearest neighbour of vector. As *d* increases, $E_1(d)$ tends to one. Embedding dimension will be given by the value *d* for which $E_1(d)$ stops to modify.

In (Cao, Mees & Judd, 1997) has been proposed a similar method to determine if the original data series is random.

Using a different metric

$$E_2(d)=\frac{E^*(d+1)}{E^*(d)}$$

where

$$E^*(d) = \frac{1}{N - d\tau} \sum_{t=1}^{N-d\tau} \left| x_{t+m\tau} - x_{t+m\tau}^{NN} \right|,$$

A random series will have $E_2(d)$ close to unity for all values of *d* while a chaotic series will have $E_2(d)$ less than one for small values of *d*.

2.4 Fractal dimension

Fractal dimension, also known as capacity dimension, Hausdorff dimension or Hausdorff-Besicovitch dimension is a dimension that allowed non integer values. The objects for which the Hausdorff dimension is different from Lebesgue covering dimension are called fractals. Hausdorff dimension of a compact metric space *X* is a real number $d_{fractal}$ such that the minimum number of open sets with diameter less than or equal to ε necessary to cover the space, $n(\varepsilon)$, is proportional with $\varepsilon^{-d_{fractal}}$ when $\varepsilon \to 0$.

Explicitly,

$$d_{\text{fractal}} \equiv \lim_{\varepsilon \to 0^+} \frac{\ln N}{\ln \frac{1}{\varepsilon}},$$

if the limit exists, where *N* is the number of elements that form the finite covering of metric space and ε is a majorant for the diameters of sets that forming the covering.

Mandelbrot estimated the Hausdorff dimension for more frontiers getting results from 1 for borders which are similar with straight lines to 1.25 for the west coast of Great Britain. Other results achieved by Mandelbrot for the Hausdorff dimension of some frontier borders were 1.15 for German border, 1.14 for the border between Spain and Portugal respectively 1.13 for Australian coast length (Mandelbrot, 1967).

There are several ways to define a fractal dimension but the most used is the correlation dimension.

Let *N* be the number of elements of the time series, *d* the embedding dimension and τ the time delay. The embedding of time series of observations in a *d*-dimensional space is achieved by building vectors

$$x_t^d = \left(x_t, x_{t+\tau}, ..., x_{t+\tau(d-1)}\right),\ t = 1,2,...,M \text{ , where } M = N - \tau(d-1).$$

Considering spheres of radius *r* around the points of embedding space, the average number of points contained in spheres, without counting their centers is given by

$$C(r) = \frac{1}{M(M-1)} \sum_{t=1}^{M} \sum_{\substack{s=1 \\ s \neq t}}^{M} H\left(r - \left\| x_t^d - x_s^d \right\|\right),$$

where x_t^d is the center of the sphere and $H(x)$ is the Heaviside function,

$$H(x) = \begin{cases} 0 \text{ daca } x < 0 \\ 1 \text{ daca } x \geq 0 \end{cases}.$$

Correlation dimension suppose that as *r* approaches zero, the relationship after that $C(r)$ changes is

$$C(r) = \lim_{r \to 0} k r^{D_C}.$$

Explaining D_C from the previous relation we obtain

$$D_C = \lim_{r \to 0} \frac{\ln C(r)}{\ln r}.$$

Because the data set is not continuous, *r* can not get too close to 0 because the spheres would not contain other points besides centers. To remove this shortcoming, in practice, we plot $\ln C(r)$ versus $\ln r$ and identify the apparently linear portion of the graph. The slope of this portion approximates the correlation dimension D_C. If D_C is integer, then the attractor is a usuall geometric object, a point for $D_C = 0$, a curve in case of $D_C = 1$ or a surface when $D_C = 2$. If D_C is not integer, then the attractor is strange and the system has a chaotic behavior (Georgescu, 2012).

2.5 Information dimension

The information function is defined by the formula

$$I = -\sum_{i=1}^{N} P_i(\varepsilon) \ln P_i(\varepsilon)$$

where $P_i(\varepsilon)$ is a natural measure or the probability that the element *i* to be populated so that

$$\sum_{i=1}^{N} P_i(\varepsilon) = 1.$$

Then, the information dimension is given by

$$d_{\text{information}} = -\lim_{\varepsilon \to 0^+} \frac{I}{\ln \varepsilon} = \lim_{\varepsilon \to 0^+} \sum_{i=1}^{N} \frac{P_i(\varepsilon) \ln P_i(\varepsilon)}{\ln \varepsilon}.$$

If each element is visited with equal probability, $P_i(\varepsilon)$ is independent of *i* and

$$\sum_{i=1}^{N} P_i(\varepsilon) = N \cdot P_i(\varepsilon) = 1.$$

Then

$$P_i(\varepsilon) = \frac{1}{N}$$

and

$$d_{\text{information}} = \lim_{\varepsilon \to 0^+} \sum_{i=1}^{N} \frac{\frac{1}{N} \ln \frac{1}{N}}{\ln \varepsilon} = \lim_{\varepsilon \to 0^+} \frac{\ln N^{-1}}{\ln \varepsilon} = -\lim_{\varepsilon \to 0^+} \frac{\ln N}{\ln \varepsilon} = d_{\text{fractal}}$$

where d_{fractal} is the fractal dimension.

The three dimensions presented satisfy the inequality

$$d_{\text{corelation}} \le d_{\text{information}} \le d_{\text{fractal}}$$

where d_{fractal} is the fractal dimension and $d_{\text{corelation}}$ is the correlation dimension.

2.6 The largest Lyapunov exponent

One of the most used techniques for determining the presence of chaotic behavior is the largest Lyapunov exponent which measures the divergence of trajectories with neighboring origins. As the system evolves distance between trajectories vary in turn.

Consider a model and two neighboring points $x_1(0)$, $x_2(0)$ at the time *t=0*, starting points for two trajectories in phase space. Denote the distance between these two points *d(0)*. At the time *t*, that is after moving the two points along the trajectories, distance between points is measured again and denoted *d(t)*.

Using a different terminology, we can say that we applied a flow Φ_t to both points and after the time period *t* we measured the distance between the two points, *d(t)*.

Is monitored the evolution of the relationship between the two distances $\frac{d(0)}{d(t)} = e^{\chi t}$.

When *t* tends to infinity, χ converges to a value. The value of this limit is Lyapunov characteristic exponent.

If $\chi > 0$, it is said that the two orbits, initially close, diverge exponentially under the action of the flow. It also says that the Lyapunov characteristic exponent measures the rate of divergence of the system (Georgescu, 2012).

Lyapunov exponent measures the rate of convergence or divergence in each dimension. A chaotic system will present the trajectory divergence at least in a dimension.

To determine the largest Lyapunov exponent is used the expression

$$\lambda_{\max} = \frac{1}{N\Delta t}\sum_{t=0}^{N-1} \ln\left(\frac{|s(t+\Delta t) - s'(t+\Delta t)|}{|s(t) - s'(t)|}\right)$$

where $s(t)$ and $s'(t)$ represent close but distinct points. As Δt grow, the Lyapunov exponent theoretically converges to its true value.

In practice, due to finite and noises data, the largest Lyapunov exponent can be determined only approximately in a range of values

After calculating the Lyapunov maximum exponent or the determination of its approximations we make assumptions about the nature of the system:

- $\lambda < 0$ The system generates a stable fixed point or a stable periodic orbit. Negative values of Lyapunov exponent are characteristic to non-conservative or dissipative systems. The higher the absolute value of the Lyapunov exponent the more stable is the system. A fixed point superstable will have a Lyapunov exponent that tends to minus infinity.
- $\lambda = 0$ A system with such an exponent is conservative.
- $\lambda > 0$ In this case the orbits are unstable and chaotic. Points initially very close diverge to arbitrary values over time. A graphical representation is similar to a cloud of points without a distinct path

3. CONCLUSIONS

In economy the majority of historical data are available as time series. Detecting chaotic nature of the processes that have provided such data is not an easy task because there is still no way to specify clearly the existence of chaos. Another constraint is the relatively small number of observations that allows us only to issue certain assumptions about the phenomenon studied and to determine estimates of chaos indicators such as largest Lyapunov exponent.

Thus in this uncertainty we can only try to highlight as many aspects that allow us cataloging process as chaotic one.

Due to these weaknesses and others such as difficulty distinguishing between deterministic chaos and noise and limited predictions to just a few steps, economists have lost the enthusiasm displayed upon discovery of chaos theory.

However, there are ideals such as guiding the economy with small impulses applied at appropriate times, to which tend theorists in economics and that could be possible using models based on chaos theory

Determination of chaotic behavior is also important to establish a correct prediction horizon.

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