A MODEL OF CONSTRUCTION OF A MINIMUM RISK PORTFOLIO BASED ON MARKOWITZ PORTFOLIO THEORY. APPLICATION ON BUCHAREST STOCK EXCHANGE

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Abstract: In this paper, the authors test a model of an efficient portfolio with minimum risk, starting from the analysis of one year portfolio payoff and risk of ten securities from Bucharest Stock Exchange. In accordance with the modern portfolio theory, maximization of returns at minimal risk should be the main objective of every investor. We show, using a mathematical methodology based on Markowitz portfolio theory and on Lagrange function, which is the exact amount of stocks to be purchased from a Bucharest Stock Exchange sample of securities in order to have an efficient portfolio with minimum risk at a given return.

JEL classification: G11, G14

Key words: efficient portfolio, risk, return, Markowitz portfolio theory, Bucharest Stock Exchange

1. INTRODUCTION

Markowitz's portfolio model is concerned with selecting optimal portfolio by risk adverse investors. According to the model, risk adverse investors should select efficient portfolios, the portfolios that maximize return at a given level of risk or minimize risk at a given level of return, which can be formed by combining securities having less than perfect positive correlations in their returns [7]. In this paper, the authors use a specific mathematical methodology, based on Markowitz portfolio theory, in order to construct an efficient portfolio, with minimum risk, given an average return, considering ten securities from Bucharest Stock Exchange.

2. THEORETICAL FRAMEWORK

Before the ideas expressed by Markowitz [4], investors were more concerned on assessing the risks and returns of individual securities in constructing their portfolios. The most common investment advice was to identify those securities that offered the best opportunities for gain with the least risk and then construct a portfolio from these. Following this advice, an investor might conclude that bank stocks were the best investment instruments when establishing a portfolio since they all offered good risk-reward characteristics. Markowitz proved that this way of thinking is not right, suggesting that the value of a security to an investor might best be evaluated by its mean, its standard deviation and its correlation to other securities in the portfolio. He warned investors that “holding securities that tend to move in concert with each other does not lower risk”[4]. Diversification, he concluded “reduces risk only when those assets are combined whose prices move inversely, or at different times, in relation to each other”[4]. This suggestion seemed to ignore all the other information concerning the company — its earnings, dividend policy, capital structure, customers and competitors —by computing relatively few simple statistics. Detailing a mathematic of diversification, he suggested that investors select their portfolios in function of their overall risk-reward characteristics instead of merely compiling portfolios from securities that each individually has attractive risk reward characteristics. In other words, investors should select portfolios not individual securities.

Harry M. Markowitz pioneered this portfolio theory in 1952 and he showed that it is possible for different portfolios to have varying levels of risk and return. Each investor must decide upon the level of risk they are ready to assume and then allocate or diversify their investment according to this decision. Investors will decide for the portfolio which better suits their risk attitude. Those who accept more risk, will select the portfolio on the efficient frontier, closer to point X, whereas the more risk adverse they are, the more closer to point Y the portfolio would be (Figure no. 1).

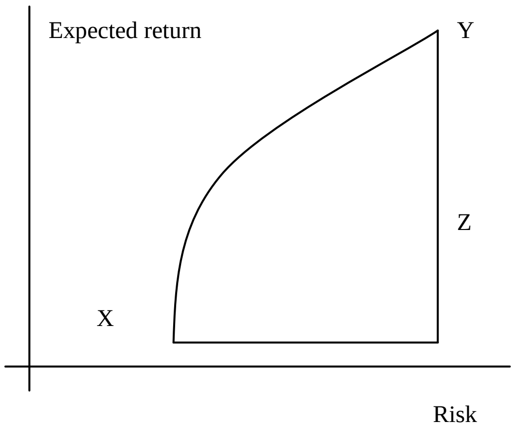

Figure no. 1. Efficient frontier

The optimal-risk portfolio is usually determined to be somewhere in the middle of the curve because as one go higher up the curve, she/he takes on proportionately more risk for a lower incremental return. But low risk/low return portfolios are pointless because she/he can achieve a similar return by investing in risk-free returns like government securities [7].

Investors can choose how much volatility she/he is willing to bear in her/his portfolio by picking any other point that falls on the efficient frontier. This will give her/him maximum return for risk s/he wishes to accept. To select a minimum variance portfolio, an investor should plot her/his indifference curves on the efficient set and then proceeds to choose the portfolio that is on the indifference curve that is farthest northeast. These portfolios will correspondence to the point at which an indifference curve is just tangent to the efficient set.

Markowitz model was theoretically elegant and conceptually sound. However, its serious limitation was the volume of work well beyond the capacity of all except a few analysts. To resolve the problem, William F. Sharpe developed a simplified variant of the Markowitz model that reduces substantially its data and computational requirements [7].

3. METHODOLOGICAL FRAMEWORK

Our analysis starts from the following hypothesis: we wish to invest a given sum (S) in k Bucharest Stock Exchange securities ($A_1, A_2 ... A_k$) at the following prices ($P_1, P_2 ... P_k$). We want to determine the exact amount to be invested in each security, in percentage ($x_1, x_2 ... x_k$) and in exact number of securities ($n_1, n_2 ... n_k$).

The next restrictions are taken into consideration:

$$x_1, x_2 ... x_k \in (0,1) \tag{1}$$

$$x_1 + x_2 + ... + x_k = 1 \tag{2}$$

We assume r the daily return obtained for each security. In this way, we will have:

$$r_{kt} = \frac{P_{kt} - P_{kt-1}}{P_{kt-1}} \tag{3}$$

where:

k – the security "k";

t – the time.

For having a minimum risk portfolio, we conduct our analysis guided by the Harry Markowitz idea: minimizing one variable while keeping the other constant, which in our case is minimizing risk by considering a given average return of the portfolio. Markowitz (1952) associated the risk with the variance in the value of a portfolio. In this way, the risk of our portfolio becomes:

$$\text{var}(rp) = \text{var}(x_1 \cdot r_1 + x_2 \cdot r_2 + ... + x_k \cdot r_{k1}) =$$

$$[x_1, x_2, ... x_n] \cdot \begin{bmatrix} \sigma^2{}_1 & \sigma_{12} & ...\sigma_{1k} \\ \sigma_{12} & \sigma^2{}_2 & ...\sigma_{2k} \\ ... & ... & ... \\ \sigma_{k1} & \sigma_{k2} & ...\sigma^2{}_k \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_k \end{bmatrix} = X^T \cdot V \cdot X \tag{4}$$

We want to minimize var (rp), considering the following restrictions:

$$x_1 + x_2 + ... + x_k = 1 \tag{5}$$

$$x_1 \cdot \bar{r}_1 + x_2 \cdot \bar{r}_2 + ... + x_k \cdot \bar{r}_k = \bar{r}_p \tag{6}$$

We also consider:

$$U = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}, R = \begin{bmatrix} \overline{r1} \\ \overline{r2} \\ \overline{r3} \\ \dots \\ \overline{rk} \end{bmatrix}, \tag{7}$$

The restrictions transform themselves into the following equations:

$$\begin{aligned} U^T \cdot X &= 1 \\ R^T \cdot X &= \bar{r}_p \end{aligned} \tag{8}$$

Using the Lagrange function, we build the following function, which must be minimized:

$$L = X^T \cdot V \cdot X + \lambda(U^T \cdot X) + \mu(R^T \cdot X - \bar{r}_p) \tag{9}$$

For these, we set the partial derivatives to zero:

$$\begin{cases} \dfrac{\partial L}{\partial x} = V \cdot X \cdot \lambda \cdot U + \mu \cdot R = 0 \\ \dfrac{\partial L}{\partial \lambda} = U^T \cdot X - 1 = 0 \\ \dfrac{\partial L}{\partial \mu} = R^T \cdot X - \bar{r}_p = 0 \end{cases} \tag{10}$$

from which we can determine X like:

$$X = -\lambda \cdot V^{-1} \cdot U - \mu \cdot V^{-1} \cdot R \tag{11}$$

where:

$$\lambda = \frac{b \cdot \overline{r_p} - c}{ac - b^2} \tag{12}$$

$$\mu = \frac{b - a \cdot \overline{r_p}}{ac - b^2} \tag{13}$$

$$a = U^T \cdot V^{-1} \cdot U \tag{14}$$

$$b = R^T \cdot V^{-1} \cdot U \tag{15}$$

$$c = R^T \cdot V^{-1} \cdot R \tag{16}$$

For the purpose of this study, we followed the trend in the price of 10 securities from Bucharest Stock Exchange that were monitored over a period of 12 months. The securities taken into consideration belong to the first tier of Bucharest Stock Exchange, being considered the most representative companies, since they have to fulfill some extremely demanding performance and transparence criteria (own funds > 30 million Euros, business plan for at least three next years etc.). On Bucharest Stock Exchange there were listed at the beginning of 2010, 21 companies at the first tier. From all these, considering their evolution in the last year, we had taken into consideration the following securities:

- *Antibiotice Iaşi (ATB)*
- *Azomureş (AZO)*
- *Biofarm (BIO)*
- *BRD Societe Generale (BRD)*
- *SSIF Broker SA (BRK)*
- *Rompetrol Rafinare (RRC)*
- *Petrom (SNP)*
- *CNTEE Transelectrica (TEL)*
- *SNTGN Transgaz SA (TGN)*
- *Banca Transilvania (TLV)*

Another reason for considering these companies in the sample was that the majority of the selected securities (more precisely 9 of the 10) make part of the BET index, which has in its composition the most liquid and traded companies on the stock market. It is true that, on the second tier, there are some companies, which in the last years, have registered greater returns, but we considered these securities not so stable, our proposed objective being that of building an efficient portfolio with minimum risk.

The analysed time-horizon is 1.01.2009-24.12.2009. Although the available data surpassed this time-horizon (we had data beginning with 2008), we considered 2008 like an atypical year, given the international financial crisis, that, through its contagion effect, lead to dramatic effects even on the Romanian capital market, with corrections of more than 70 % of the market indexes.

4. RESULTS

For each share it was known the average return of R_i, in this way resulting an average return of portfolio of 0,002984 (aprox. 3 %) for the analysed period of time. With the help of Lagrange function (Eq.), we can determine exactly the percentage to be invested in every share (X), in order t keep a minimum risk. The table below (Table no.1) presents the obtained results:

Table no. 1. Percentage of the available sum to be invested in every share of the portfolio.

Share	Percentage
ATB	0.115768 (11,58 %)
AZO	0.069131 (6,91 %)
BIO	0.070171 (7,02 %)
BRD	0.065188 (6,52 %)
BRK	0.056678 (5,67l%)
RRC	0.162279 (16,23 %)
SNP	0.077519 (7,75 %)
TEL	0.049572 (4,96 %)
TGN	0.194917 (19.49 %)
TLV	0.138777 (13,88 %)

Further we have computed how many shares could have been purchased at the start of the time period (Table no. 2), with the sum of 10.000 RON, with the help of the subsequent formula:

$$n_{i0} = \frac{x * S_0}{P_{i0}} \tag{17}$$

where: n = no. of shares "i";
x = percent of the invested sum in the share "i";
p = trade price of the share „i" at the moment 0.

Table no.2. Volumes of shares to be purchased.

Shares	Volume of shares to be purchased	Trade price 4.01.2010	Sum to be invested in every share	Adjusted volume of shares to be purchased
ATB	3047	0.625	1904.081	1607
AZO	4431	0.288	1276.257	2338
BIO	8589	0.202	1734.946	4531
BRD	78	13.2	1024.386	41
BRK	5668	0.205	1161.897	2990
RRC	76547	0.0646	4944.913	40384
SNP	4123	0.252	1039.089	2175

TEL	44	13	570.3013	23
TGN	16	159	2561.307	8
TLV	1262	2.17	2737.687	666
			18954.86	

At the level of 4.01.2010 trade prices, if we purchase the indicated volumes of shares, the average invested sum would be of 18954.86, and not 10.000 RON. This is the reason why, in accordance with the proportionality $\frac{S_0}{S_1} = \frac{10.000}{18954.86} = 0.527569$ we would only purchase:

- 1607 shares ATB
- 2338 shares AZO
- 4531 shares BIO
- 41 shares BRD
- 2990 shares BRK
- 40.384 shares RRC
- 2175 shares SNP
- 23 shares TEL
- 8 shares TGN
- 666 shares TLV

5. Conclusions

Starting from the Markowitz portfolio theory, we have tested a model of an efficient portfolio, which offers minimum risk, at a given return, with application on a Bucharest Stock Exchange sample. In order to do this, we had been previously monitorizing over a period of one year the returns and risks associated with the 10 securities from Bucharest Stock Exchange, assuming that the the past can provide some information on how the returns will behave in the future. The mathematical computations offered us the exact amount of stocks to be purchased from the 10 securities belonging to Bucharest Stock Exchange, in order to obtain the desired efficient portfolio.

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